§9. Two-dim Chiral QFT - I

<u>Recall</u>: 1st order formalism of Top. QM

$$\text{fields} = \Omega^{\bullet}(S^1, V) \quad \text{w/. de Rham differential}$$

de Rham being part of the BRST operator implies that
"translation is homologically trivial"

$\Longrightarrow$ top. theory.

We will now consider 2d Chiral models where

$$\text{fields} = \Omega^{0,*}(\Sigma, h) \quad \text{w/. Dolbeault diff. } \bar\partial$$

$\bar\partial$ being part of the BRST operator, implies that
"anti-hol. is homological trivial"

$\Longrightarrow$ Chiral (holomorphic) theory.

· In top. QM, the theory is UV finite, we find that the renormalization process is "smart":

$L=0$: "$dI + \hbar \Delta I + \frac{1}{2}\{I,I\} = 0$" ill-defined

$$\Big\downarrow \; e^{\frac{1}{\hbar}I[L]} = \lim_{\varepsilon\to 0} e^{\hbar P_\varepsilon^L} e^{\frac{1}{\hbar}I} \quad \text{exists}$$

$L>0$: $dI[L] + \hbar \Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0$ well-defined QME

$$L\to 0 \Big\downarrow \; \text{the meaning of this eqn}$$

$L=0$: $dI + \frac{1}{2\hbar}[I,I] = 0$ QME at local

Moyal-Weyl Commutator

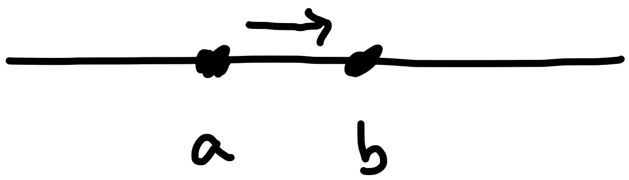
In particular, we find QME = Fedosov equation.

· We will see that 2d Chiral theory is also UV finite and we have a similar geometric result for QME

Ref: S. Li: Vertex algebras and quantum master equation

• Vertex algebra

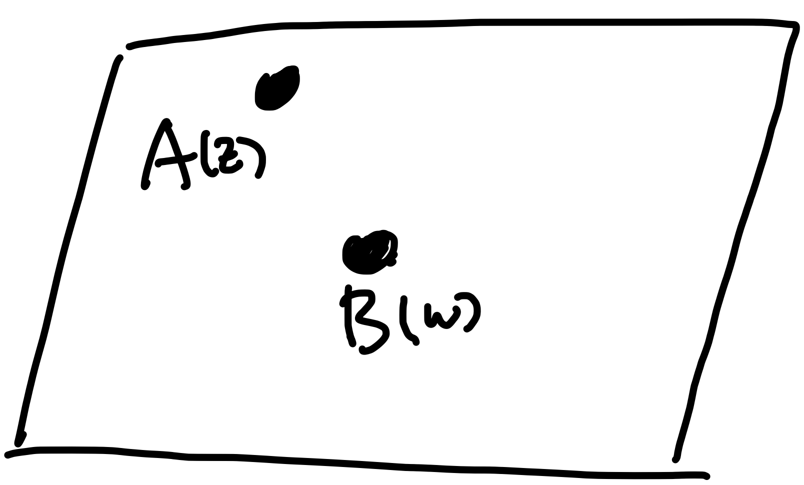
In 1d top. theory



associative algebra

$a \cdot b$

In 2d chiral theory



(chiral) vertex algebra

$$A(z)B(w) \sim \sum_n \frac{(A_{(n)}B)(w)}{(z-w)^{n+1}}$$

The "product" depends on the location holomorphically.

$\Rightarrow$ ∞-many binary operations

Def'n: A vertex algebra is a collection of data

- (Space of states) a $\mathbb{Z}_2$-graded superspace $V = V_0 \oplus V_1$
- (vacuum) a vector $|0\rangle \in V_0$
- (translation operator) an even linear map $T : V \to V$
- (state-field correspondence) an even linear operation

$$Y(-,z) : V \longmapsto \mathrm{End}\, V\, [[z, z^{-1}]] \quad \text{(vertex operation)}$$

$$A \longmapsto Y(A,z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$$

Such that $Y(A,z)B \in V((z))$ for any $A, B \in V$.

They are required to satisfying the following axiom:

- (vacuum axiom) $Y(|0\rangle, z) = 1_{V}$. For any $A \in V$,

$$Y(A,z)|0\rangle \in V[[z]] \text{ and } \lim_{z \to 0} Y(A,z)|0\rangle = A.$$

- (translation axiom) $T|0\rangle = 0$. For any $A \in V$,

$$[T, Y(A,z)] = \partial_z Y(A,z)$$

- (locality axiom) All $\{Y(A,z)\}_{A \in V}$ are mutually local.

Roughly speaking, mutually locality implies that for any $A, B \in V$, we can expand as

$$Y(A,z)Y(B,w) = \sum_{n\in\mathbb{Z}} \frac{Y(A_{(n)}\cdot B, w)}{(z-w)^{n+1}}$$

This is called operator product expansion (OPE)

$\{A_{(n)}\cdot B\}$ from the expansion coefficient can be viewed as defining ∞ tower of products.

For simplicity, we will simply write

$$A(z) \equiv Y(A,z) \text{ for } A \in V.$$

Then the OPE is written as

$$A(z)B(w) = \sum_{n\in\mathbb{Z}} \frac{A_{(n)}\cdot B\,(w)}{(z-w)^{n+1}}$$

and we also write

$$A(z)B(w) \sim \sum_{n\geq 0} \frac{A_{(n)}\cdot B(w)}{(z-w)^{n+1}}$$

Singular part.

Given a vertex algebra, we can define its modes Lie algebra $\oint V$ as follows:

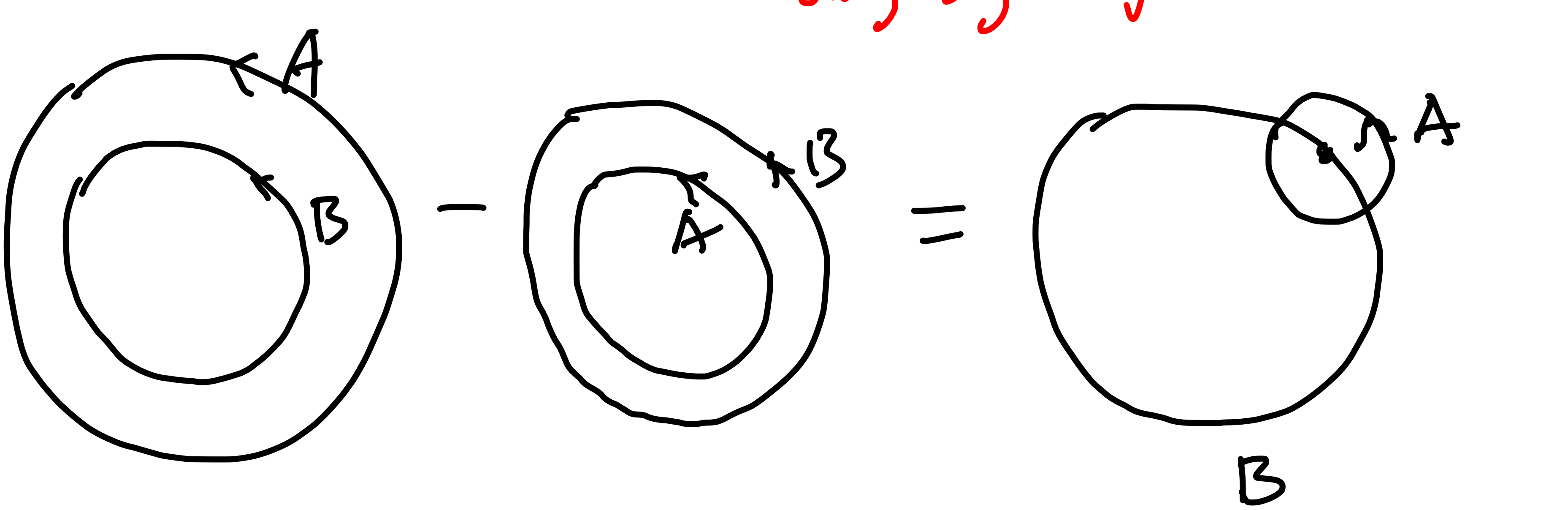
$$\oint V := \mathrm{Span}_{\mathbb{C}} \left\{ \oint dz\, z^k A(z) = A_{(k)} \right\}_{\substack{A\in V \\ k\in\mathbb{Z}}}$$

The Lie bracket is determined by the OPE

$$\left[\oint dz\, z^m A(z) ,\ \oint dw\, w^n B(z) \right]$$

$$= \oint dw\, w^n \oint_w dz\, z^m \sum_{j\in\mathbb{Z}} \frac{A_{(j)}\cdot B(w)}{(z-w)^{j+1}}$$

only singular part matters here



Example: $\beta\gamma$ - system

This is generated by two bosonic fields $\beta(z), \gamma(z)$ w/.

$$\beta(z)\gamma(w) \sim \frac{\hbar}{z-w} \sim -\gamma(z)\beta(w)$$

The vertex algebra $\mathcal{V}$ is identified w/. diff. ring

$$\mathcal{V} =: \mathbb{C}[[\partial^i\beta, \partial^i\gamma]]: [[\hbar]]$$

and the general OPE is obtained via Wick Contractions

For example,

$$:\beta(z)\gamma(z): \ :\beta(w)\gamma(w):$$

$$= \frac{\hbar}{z-w} :\gamma(z)\beta(w): - \frac{\hbar}{z-w} :\beta(z)\gamma(w):$$

(1 - contraction)

$$+ \left(\frac{\hbar}{z-w}\right)^2 \quad \text{(2- contraction)}$$

$$= \sum_{k\geq 0} \frac{\hbar}{z-w} \frac{(z-w)^k}{k!} :\partial^k\gamma(w)\beta(w) - \partial^k\beta(w)\gamma(w):$$

$$+ \frac{\hbar^2}{(z-w)^2}$$

Example: bc-system

This is generated by two fermionic fields $b(z), c(z)$ w/.

$$b(z)c(w) \sim \frac{\hbar}{z-w} \sim c(z)b(w)$$

The vertex algebra is given by the diff. ring

$$\mathcal{V} = \,:\mathbb{C}[[\partial^i b, \partial^i c]]:[[\hbar]]$$

The general OPE is generated in the similar way (need to take care of the signs).

More generally, we can define a general $\beta\gamma$-bc system by considering a $\mathbb{Z}_2$-graded space $h = h_0 \oplus h_1$

w/. an even symplectic pairing

$$\langle -,- \rangle : \Lambda^2 h \longmapsto \mathbb{C}$$

Let $\{a_i\}$ be a basis of h, then we can define a vertex algebra V_h by

$$V_h = :\mathbb{C}[[\partial^k a_i]]:\ [[\hbar]]$$

The OPE is generated by

$$a_i(z)\, a_j(w) \sim \frac{\hbar \langle a_i, a_j \rangle}{z-w}$$

In particular,

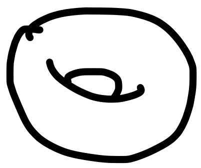
$$h_0 \rightsquigarrow \text{copies of } \beta\gamma\text{-system}$$

$$h_1 \rightsquigarrow \text{copies of } bc\text{-system}$$

In the next, we will mainly focus on $\beta\gamma - bc$ systems.

- Chiral deformation of $\beta\gamma$–bc systems

We consider the following data:

$E =$  elliptic curve, z linear coord. $z \sim z+1 \sim z+\tau$

$h = h_0 \oplus h_1$, $\langle -,- \rangle$: graded symplectic space as above

This defines a field theory in BV formalism by:

fields: $\mathcal{E} = \Omega^{0,*}(E) \otimes h$

(-1)-symplectic pair

$$\omega(\varphi_1, \varphi_2) = \int_E dz \wedge \langle \varphi_1, \varphi_2 \rangle \qquad \varphi_i \in \mathcal{E}$$

Note that ω has $\deg = -1$. Since we need exactly 1 $\overline{dz}$ from φ_1, φ_2 to be integrated.

The free theory is given by

$$\frac{1}{2}\int_E dz \langle \varphi, \bar{\partial}\varphi \rangle \qquad \varphi \in \mathcal{E}$$

The local quantum observables form exactly $\beta\gamma$-bc system.

The propagator is given by Szego kernel

$$\bar{\partial}^{-1} \sim \frac{1}{z-w} + \text{regular}$$

We would like to consider a general interacting theory by turning on chiral deformations

$$\int \mathcal{L}(\varphi, \partial_z \varphi, \partial_z^2 \varphi, \cdots)$$

which involves only holomorphic derivatives.

This is related precisely to the vertex algebra $\mathcal{V}_{h^\vee} = \mathbb{C}[[\partial^i h^\vee]][[\hbar]]$ as follows

Define $I : \mathcal{V}_{h^\vee} \longmapsto \mathcal{O}_{loc}(\mathcal{E})$

$$\gamma \longmapsto I_\gamma$$

Explicitly, if $\gamma = \sum \partial^{k_1} a_1 \cdots \partial^{k_m} a_m$, then

$$I_\gamma(\varphi) = i \int_E dz \sum \pm \, \partial_z^{k_1} a_1(\varphi) \cdots \partial_z^{k_m} a_m(\varphi)$$

Here $a_i \in h^\vee$, $a_i(\varphi) \in \Omega^{0,*}(E)$.

Thm [UV finiteness] For any $\gamma \in \mathcal{V}_{h^\vee}$, the chiral deformed theory

$$\frac{1}{2} \int_E dz \langle \varphi, \bar\partial \varphi \rangle + I_\gamma(\varphi)$$

is UV finite in the sense that

$$e^{\frac{1}{\hbar} I_\gamma[L]} := \lim_{\varepsilon \to 0} e^{\hbar P_\varepsilon^L} e^{\frac{1}{\hbar} I_\gamma} \text{ exists.}$$

Rk: The proof is a bit technical. See the reference.
The reason is different from Top. QM, where we see that the propagator is bounded (though not continuous). Here the graph integral is <u>NOT</u> absolute convergent. In the next lecture, I will give a geometric interpretation of this fact.

Once we have a well-defined $I_\gamma[L]$ described above, we can formulate the effective QME

$$\partial I_\gamma[L] + \hbar \Delta_L I_\gamma[L] + \frac{1}{2}\{I_\gamma[L], I_\gamma[L]\}_L = 0$$

and ask for the condition of γ to satisfy this equation. It turns out that the answer is very simple.

Thm [L] Let $\gamma \in \mathcal{V}_{\mathfrak{h}^\vee}$ and $I_\gamma[L]$ the effective functional defined above via the UV finiteness.
Then $I_\gamma[L]$ satisfies the effective QME

$$\bar\partial I_\gamma[L] + \hbar \Delta_L I_\gamma[L] + \frac{1}{2}\{I_\gamma[L], I_\gamma[L]\} = 0$$

if and only if

$$[\oint \gamma, \oint \gamma] = 0 \quad \text{in } \oint \mathcal{V}.$$

RK: The local quantum observable of the chiral deformed theory is the vertex algebra

$$H^\bullet(\mathcal{V}_{\mathfrak{h}^\vee}, \oint \gamma)$$

So $\oint \gamma$ plays the role of BRST reduction.
Reversely, vertex algebras coming from the BRST reduction of free field realizations can be realized via the model of chiral deformations above.

The above theorem can be glued for a chiral σ-model

$$\varphi : E \longmapsto X$$

which produces a bundle $\mathcal{V}(X)$ of chiral vertex algebras $\mathcal{V}(X)$ on X.

$$\begin{array}{c} \mathcal{V}(X) \\ \downarrow \\ X \end{array}$$

Then the sol'n of effective QME asks for a flat connection on $\mathcal{V}(X)$ of the form

$$D = d + \frac{1}{\hbar}\left[\oint \gamma, -\right] \qquad D^2 = 0.$$

Here $\gamma \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \gamma$ is fiberwise chiral mode operator. This can be viewed as the chiral analogue of Fedosov connection.